S.-T. Yau College Student Mathematics Contests 2023

Analysis and Differential Equations

Team

Please solve two from the following three problems.

1. Given two lattices $\Gamma_1 = \mathbb{Z} \cdot \omega_{1,1} + \mathbb{Z} \cdot \omega_{1,2}$ and $\Gamma_2 = \mathbb{Z} \cdot \omega_{2,1} + \mathbb{Z} \cdot \omega_{2,2}$ in the complex plane $\mathbb{C}$ having full rank. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function inducing a holomorphic map between two elliptic curves $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_2$, i.e.,

$$f(z) = f(z + \gamma) \pmod{\Gamma_2} \qquad (\forall\, z \in \mathbb{C}, \gamma \in \Gamma_1).$$

Show that $f(z) = a \cdot z + b$ for any $z \in \mathbb{C}$, where a, b are some complex numbers satisfying $a \cdot \Gamma_1 \subset \Gamma_2$, $b \in \Gamma_2$.

2. Let T be a bounded self-adjoint operator, prove that the spectrum of T is not empty. If T is self-adjoint but unbounded, is the spectrum of T necessarily non-empty? Prove your conclusion.

3. Let $\{f_n\}_n$ be a bounded sequence in $L^2(\mathbb{R}^3)$, and let $\{g_n\}_n$ be a sequence of Schwartz functions.

Assume,

$$(-\Delta + |x|^2) g_n = f_n. \tag{0.1}$$

Prove that

- $\{g_n\}_n$ is a bounded sequence in $L^2(\mathbb{R}^3)$.
- One can find a subsequence $\{g_{n_k}\}_k$, which converges in $L^2(\mathbb{R}^3)$.